Talk at Yale

Shuffle realization of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$
and Bethe subalgebras of $U_q(\widehat{\mathfrak{gl}}_n)$

April 14th, 2015

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(1.1) The quantum toroidal algebra of gl_1: $\ddot{U}_{q_1,q_2,q_3}(gl_1)$

Though the main results of today's talk concern the classical quantum toroidal algebra of sl_n, we will spend the first half of this talk on the "baby analogue": $\ddot{U}_{q_1,q_2,q_3}(gl_1)$.

This algebra became of big interest as it has several different algebraic incarnations and also appears naturally in geometry.

Def-n: Choose $q_1, q_2, q_3 \in \mathbb{C}^*$ s.t. $q_1 \cdot q_2 \cdot q_3 = 1$.

The algebra $\ddot{U}_{q_1,q_2,q_3}(gl_1)$ is generated by

$$\{e_i, f_i, \psi_i, \psi_0^{-1}, \gamma^{\pm 1/2}\}$$

with the following defining relations:

- $\gamma^{\pm 1/2}, \psi_0^{\pm 1}$ – central and $[\psi^{\pm}(z), \psi^{\pm}(w)] = 0$.
- $e(z)e(w) = g\left(\frac{z}{w}\right) e(w)e(z)$
- $f(z)f(w) = g\left(\frac{w}{z}\right) f(w)f(z)$
- $\psi^{\pm}(z)e(w) = g\left(\gamma^{\pm 1/2}\cdot\frac{z}{w}\right) e(w)\psi^{\pm}(z)$
- $\psi^{\pm}(z)f(w) = g\left(\gamma^{\pm 1/2}\cdot\frac{w}{z}\right) f(w)\psi^{\pm}(z)$
- $[e(z), f(w)] = \frac{1}{\varkappa_1}\cdot\left\{\delta\left(\frac{\gamma w}{z}\right)\psi^{+}(\gamma^{1/2}w) - \delta\left(\frac{\gamma z}{w}\right)\psi^{-}(\gamma^{1/2}z)\right\}$; $\varkappa_1 := (1-q_1)(1-q_2)(1-q_3)$
- $\mathrm{Sym}_{\mathfrak{S}_3} \frac{z_2}{z_3}\cdot[e(z_1),[e(z_2),e(z_3)]] = 0 = \mathrm{Sym}_{\mathfrak{S}_3} \frac{z_2}{z_3}[f(z_1),[f(z_2),f(z_3)]]$.
- $g\left(\gamma^{-1}\frac{z}{w}\right)\psi^{+}(z)\psi^{-}(w) = g\left(\gamma\frac{z}{w}\right)\psi^{-}(w)\psi^{+}(z)$, where

$$\boxed{g(t) := \frac{(1-q_1t)(1-q_2t)(1-q_3t)}{(1-q_1^{-1}t)(1-q_2^{-1}t)(1-q_3^{-1}t)}}$$

and

$$\boxed{e(z) := \sum_{i\in\mathbb{Z}} e_i z^{-i},\quad f(z) := \sum_{i\in\mathbb{Z}} f_i z^{-i},\quad \psi^{\pm}(z) := \psi_0^{\pm 1} + \sum_{\pm i>0} \psi_i\cdot z^{-i},\quad \delta(z) := \sum_{i\in\mathbb{Z}} z^i}$$

Rmks: (1) One can also add extra "degree generators" D_1, D_2 (we will spell out a similar construction for sl_n latter on).

(2) This algebra is defined similarly to $U_q(\widehat{sl}_2)$, main difference in f-n $g(t)$.

(1.2) Geometric Motivation for $\ddot{U}_{q_1,q_2,q_3}(gl_1)$

Recall the Nakajima's construction of the Heisenberg algebre action on the space $\bigoplus_{n\geq 0} H^{\mathbb{C}^*\times\mathbb{C}^*}((\mathbb{C}^2)^{[n]})_{loc}$. Set $X := \mathbb{C}^2$. Main tool is:

$$Z[n;i] \subset X^{[n]} \times X^{[n+i]}, \qquad X^{[n]} \xleftarrow{p_{n,i}} Z[n;i] \xrightarrow{q_{n,i}} X^{[n+i]}$$

This *incidence correspondence* $Z[n;i]$ parametrizing $\{(I_1, I_2) \mid I_2 \subset I_1,\ \mathrm{supp}(I_1/I_2) = \{\cdot\}\}$

Nakajima: Heisenberg generators $\{a_{\pm i}\}_{i=1}^{\infty}$ act via

$$\bigoplus_{n\geq 0} q_{n,i*}\, p_{n,i}^* \quad \text{and} \quad \bigoplus_{n\geq i} p_{n,i*}\, q_{n,i}^*$$

Q-n: Motivated by the general principle, we should have a similar algebre (Heisenberg itself or some deformation of it) acting on $\boxed{\bigoplus_{n\geq 0} K^{\mathbb{C}^*\times\mathbb{C}^*}(X^{[n]})_{loc} =: M}$

Issue: The correspondence $Z[n,i]$ is highly singular for $i \gg 1$ which makes computations much worse.

Solution: We will utilize only $Z[n;1]$ which is non-singular, together with the tautological line bundle $\mathcal{L}$ over it.

$$\text{fiber } \mathcal{L}_{(I_1,I_2)} = I_1/I_2$$

Define the operators

$$\boxed{\begin{array}{l} e_i := \bigoplus_{n\geq 0} q_{n,1*}\left(p_{n,1}^*(-)\otimes \mathcal{L}^{\otimes i}\right), \quad f_i := \bigoplus_{n\geq 1} p_{n,1*}\left(q_{n,1}^*(-)\otimes \mathcal{L}^{\otimes(i-1)}\right) \\ \psi^{\pm}(z) := -\left(\frac{1-t_1^{-1}t_2^{-1}z^{-1}}{1-z^{-1}}\cdot c(z)\right)^{\pm}, \quad c(z) := \frac{a(t_1 z)\,a(t_2 z)\,a(t_1^{-1}t_2^{-1}z)}{a(t_1^{-1}z)\,a(t_2^{-1}z)\,a(t_1 t_2 z)}, \quad a(w) := [\Lambda^{\bullet}_{-1/w}(\mathcal{T})] \end{array}}$$

where $\mathcal{T}$ - tautological v. bundle over $\oplus X^{[n]}$, $(\ldots)^{\pm}$ - expansion in $z^{\mp 1}$.

Thm [FT, SV]: These operators define an action of

$$\ddot{U}_{t_1,t_2,\frac{1}{t_1 t_2}}(gl_1) \curvearrowright M \quad \text{with } \gamma^{\pm 1/2} = Id_M. \quad \left(\begin{array}{c} t_1, t_2 \text{ - equivariant} \\ \text{parameters of} \\ \mathbb{C}^*\times\mathbb{C}^* \end{array}\right)$$

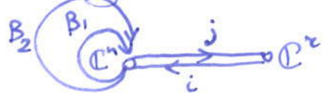
(1.2) Higher rank: realization via $M(r,n)$

Recall that $(\mathbb{C}^2)^{[n]}$ has a natural "higher rank" analogue.

Def: $M(r,n) :=$ Gieseker space of rank r and $c_2 = n$ torsion free sheaves on $\mathbb{P}^2$, whose $\mathbb{C}$-points parametrize isom. classes

$$\left\{ (E,\Phi) \;\middle|\; \begin{array}{l} E\text{-torsion free sheaf on } \mathbb{P}^2,\ rk(E)=r,\ c_2(E)=n,\ \text{loc. free in nbhd of } l_\infty \\ \Phi: E_{|l_\infty} \xrightarrow{\sim} \mathcal{O}^{\oplus r}_{|l_\infty} \text{ - trivialization at } l_\infty = \{(0:*:*)\} \subset \mathbb{P}^2 \end{array} \right\}$$

Rmks: (1) It is the simplest example of Nakajima quiver variety corresponding to the quiver



(2) For $r=1$, we have $M(1,n) \simeq (\mathbb{C}^2)^{[n]}$.

(3) There is a natural action of $\mathbb{T}_r := (\mathbb{C}^*)^2 \times (\mathbb{C}^\times)^r \curvearrowright M(r,n)$ and the fixed locus $M(r,n)^{\mathbb{T}_r}$ is parametrized by r-partitions $\bar{\lambda} = (\lambda^1, \ldots, \lambda^r)$ of n.

(4) There is an analogue of $Z[n,1]$, called the Hecke correspond.

$$M(r;n,n+1) \subset M(r;n) \times M(r;n+1)$$

which parametrize

$$\left\{ \begin{array}{l} (B_1^{(n+1)}, B_2^{(n+1)}, i^{(n+1)}, j^{(n+1)}) \text{ - quiver data} \\ \qquad + \\ \text{1-dim subspace } S \subset \operatorname{Ker} j^{(n+1)} \text{ which is } B_1^{(n+1)}, B_2^{(n+1)}\text{-invariant} \end{array} \right\} \Big/ GL_{n+1}(\mathbb{C}).$$

It is naturally equipped with the line bundle $\mathcal{L}_r$.

Let $p_n^{(r)}, q_n^{(r)}$ be the natural projections of $M(r;n,n+1)$ to 1st & 2nd components. Define:

$$e_i := \bigoplus_{n\geq 0} q^{(r)}_{n*}\left(p_n^{(r)*}(-) \otimes \mathcal{L}_r^{\otimes i}\right), \qquad f_i := \bigoplus_{n\geq r} p^{(r)}_{n*}\left(q_n^{(r)*}(-) \otimes \mathcal{L}_r^{\otimes (i-r)}\right)$$

$$\psi^{\pm}(z) := \left((-1)^r t_1 t_2 \chi_1 \ldots \chi_r \prod_{a=1}^{r} \frac{1-t_1t_2\chi_a z}{1-\chi_a z} \cdot C_r(z)\right)^{\pm}$$

where $\{t_1, t_2, \chi_1, \ldots, \chi_r\}$ - equiv. parameters of $\mathbb{T}^r$; $C_r(z)$ is defined analogously to $r=1$.

Thm: These operators define an action of $\ddot{U}_{t_1,t_2,\frac{1}{t_1t_2}}(gl_1) \curvearrowright \bigoplus_{n\geq 0} K^{\mathbb{T}_r}(M(r,n))_{loc}$ with $\gamma^{\pm 1/2} = Id$.

1.3 Representations $V(u)$ and $\mathcal{F}(u)$

Once the algebra $\ddot{U}_{q_1,q_2,q_3}(gl_1)$ appeared geometrically, it also became a subject of interest from an algebraic point of view.

Certain classes of $\ddot{U}_{q_1,q_2,q_3}(gl_1)$ were studied by

Feigin - Feigin - Jimbo - Miwa - Mukhin in arxiv: 1002.3100, 1002.3113, 1110.5310

We will recall the two interesting classes.

- Vector representation $V(u)$

  - $V(u)$ has a basis $\{[u]_i \mid i \in \mathbb{Z}\}$ (here $u \in \mathbb{C}^*$)
  - The $\ddot{U}_{q_1,q_2,q_3}(gl_1)$ - action is given via:

$$\begin{cases} e(z)[u]_i = \frac{1}{1-q_1}\cdot \delta\left(\frac{q_1^i u}{z}\right)\cdot [u]_{i+1} \\ f(z)[u]_{i+1} = -\frac{1}{1-q_1^{-1}}\cdot \delta\left(\frac{q_1^i u}{z}\right)\cdot [u]_i \\ \psi^{\pm}(z)[u]_i = \left(\frac{(z-q_1^i q_2 u)(z-q_1^i q_3 u)}{(z-q_1^i u)(z-q_1^{i-1}u)}\right)^{\pm}\cdot [u]_i \\ \gamma^{\pm 1/2}[u]_i = [u]_i \end{cases}$$

Rmks: (1) $\forall u \in \mathbb{C}^*$ $V(u)$ is obtained from $V(1)$ via the twist by an automorphism $\phi_u \in \mathrm{Aut}(\ddot{U}_{q_1,q_2,q_3}(gl_1))$

(2) The representations $V(u)$ are not in a "reasonable category $\mathcal{O}$".

- Formal coproduct

All the reps-s in above papers have $\gamma^{\pm 1/2} = 1 \Rightarrow$ let $\ddot{U}'(gl_1) := \ddot{U}(gl_1)/(\gamma^{\pm 1/2}-1)$.

The algebra $\ddot{U}'(gl_1)$ has a formal coproduct structure:

$$\Delta: e(z) \mapsto e(z)\otimes 1 + \psi^-(z)\otimes e(z),\quad f(z)\mapsto f(z)\otimes\psi^+(z) + 1\otimes f(z),\quad \psi^{\pm}(z)\mapsto \psi^{\pm}(z)\otimes\psi^{\pm}(z)$$

"Formal": $\Delta(e_i)$ and $\Delta(f_i)$ involve infinitely many terms

However, as we will see in a moment, one can make sense of $V_1 \otimes V_2$ in many cases.

1.3 Representations $V(u)$ and $\mathcal{F}(u)$

- The representation $V(u_1)\otimes V(u_2)$

Consider the natural basis $\{[u_1]_{k_1}\otimes[u_2]_{k_2} \mid k_1,k_2\in\mathbb{Z}\}$ of the space $V(u_1)\otimes V(u_2)$.
We want to make sense of

$$\{e(z)\otimes 1+\psi^-(z)\otimes e(z)\}([u]_{k_1}\otimes[u_2]_{k_2})$$

The first term is well-defined, while the second term equals

$$\left(\frac{(z-q_1^{k_1}q_2u_1)(z-q_1^{k_1}q_3u_1)}{(z-q_1^{k_1}u_1)(z-q_1^{k_1-1}u_1)}\right)^-[u_1]_{k_1}\otimes\frac{1}{1-q_1}\delta\left(\frac{q_1^{k_2}u_2}{z}\right)[u_2]_{k_2+1}.$$

BUT: We have an equality $\boxed{G(z)\cdot\delta\left(\frac{z}{w}\right)=G(w)\,\delta\left(\frac{z}{w}\right)}$ $\forall$ rational f-n $G(\cdot)$.

In particular, the above becomes:

$$\frac{1}{1-q_1}\cdot\frac{(q_1^{k_2}u_2-q_1^{k_1}q_2u_1)(q_1^{k_2}u_2-q_1^{k_1}q_3u_1)}{(q_1^{k_2}u_2-q_1^{k_1}u_1)(q_1^{k_2}u_2-q_1^{k_1-1}u_1)}\cdot\delta\left(\frac{q_1^{k_2}u_2}{z}\right)\cdot[u_1]_{k_1}\otimes[u_2]_{k_2+1}$$

This is well-defined if $u_1/u_2\notin q_1^{\mathbb{Z}}$

- Fock representation $\mathcal{F}(u)$

By above we get $\ddot{U}'(gl_1)\curvearrowright V(u)\otimes V(q_2^{-1}u)\otimes\dots\otimes V(q_2^{1-N}u)=:V^{(N)}(u)$.
Consider $W^{(N)}(u)\subset V^{(N)}(u)$ spanned by

$$\{|\lambda\rangle_u=[u]_{\lambda_1}\otimes[uq_2^{-1}]_{\lambda_2-1}\otimes\dots\otimes[uq_2^{1-N}]_{\lambda_N-N+1} \mid \lambda_1\geq\lambda_2\geq\dots\geq\lambda_N\}$$

Then $\ddot{U}'(gl_1)\curvearrowright W^{(N)}(u)$, but $\ddot{U}'(gl_1)\not\curvearrowright W^{(N,+)}(u)$ (given by additional condition $\lambda_N\geq 0$)

However: One can define the $\ddot{U}'(gl_1)$-action on the limit $\lim\limits_{N\to\infty}W^{(N,+)}(u)$ with basis parametrized by Young diagrams.

Lemma: (1) $\mathcal{F}(1)\simeq M$ – geom. representation coming from $(\mathbb{C}^2)^{[n]}$.

(2) $\mathcal{F}(\chi_1)\otimes\dots\otimes\mathcal{F}(\chi_n)$ is well-defined for generic $\{\chi_i\}$ and $\simeq$ geom. representation coming from $M(r,n)$.

- Macmahon modules

Applying the same construction, but starting from $\mathcal{F}(u)$ rather than $V(u)$ we get repr-n with basis parametrized by plane partitions (see arXiv: 1110.5310)

2.1 Small shuffle algebra S^{sm}

Historical Comment

Shuffle algebras we first introduced and studied by Feigin-Odesskii in late 90's. These algebras depend on an elliptic curve $\mathcal{E}$ and two automorphisms τ_1, τ_2 of $\mathcal{E}$ (therefore the name "elliptic shuffle alg-s").
Degenerating $(\mathcal{E}, \tau_1, \tau_2)$ into ($\mathbb{C}P^1$ with a double point, $q_1 \in \mathbb{C}^*$, $q_2 \in \mathbb{C}^*$) we "get" the small shuffle algebra.

Since we can't make precise this "degeneration procedure", we will just start from the definition of S^{sm}.

- Consider an ambient $\mathbb{N}$-graded vector space $\mathbb{S}^{sm} = \bigoplus_{n \geq 0} \mathbb{S}_n^{sm}$ with $\mathbb{S}_n^{sm} = \{\mathfrak{S}_n\text{-symmetric rational functions in } x_1, \ldots, x_n\}$.
- Define the $\overset{sm}{\star}$-product on $\mathbb{S}^{sm}$ via $\mathbb{S}_k^{sm} \times \mathbb{S}_\ell^{sm} \xrightarrow{\overset{sm}{\star}} \mathbb{S}_{k+\ell}^{sm}$

$$(F \overset{sm}{\star} G)(x_1, \ldots, x_{k+\ell}) := \mathrm{Sym}_{\mathfrak{S}_{k+\ell}} \left(F(x_1, \ldots, x_k) G(x_{k+1}, \ldots, x_{k+\ell}) \prod_{i \leq k}^{j > k} \lambda(x_i/x_j) \right)$$

where $\lambda(t) := \frac{(q_1 t - 1)(q_2 t - 1)(q_3 t - 1)}{(t-1)^3}$ (as before $q_1 q_2 q_3 = 1$)

- $\mathbb{S}^{sm}$ is *too big* ⇝ consider a graded subspace $S = \bigoplus_n S_n$ given by:
  (1) Pole conditions: $F(x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n)}{\prod_{i \neq j} (x_i - x_j)}$, $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{\mathfrak{S}_n}$.
  (2) Wheel conditions: $F(x_1, \ldots, x_n) = 0$ if $x_1/x_2 = q_i$, $x_2/x_3 = q_j$, $1 \leq i \neq j \leq 3$

Lemma: The space S is closed under $\star$-product

Def: The algebra $(S, \overset{sm}{\star})$ is called the *small shuffle algebra*

Lemma: The map $e_i \mapsto x^i$ extends to a homomorphism

$$\ddot{U}_{q_1,q_2,q_3}(gl_1)^+ \xrightarrow{\Psi_1} S.$$

(generated just by $\{e_i\}$)

Thm (Negut: arXiv 1209.3349): Ψ_1 is an isomorphism.

2.2 Commutative subalgebra $\mathcal{A}^{sm} \subset S^{sm}$

We will now recall an interesting construction, due to

Feigin – Hashizume – Hoshino – Shiraishi – Yanagida, arXiv: 0904.2291

Consider an $\mathbb{N}$-graded vector subspace $\mathcal{A}^{sm} = \bigoplus_{n \geq 0} \mathcal{A}^{sm}_n$ of S^{sm}:

$$\mathcal{A}^{sm}_n := \{F \in S^{sm}_n \mid \partial^{(0;k)} F = \partial^{(\infty;k)} F \quad \forall\, 0 \leq k \leq n\}$$

where

$$\partial^{(0;k)} F := \lim_{\xi \to 0} F(x_1, \ldots, x_{n-k}, \xi \cdot x_{n-k+1}, \ldots, \xi \cdot x_n)$$

$$\partial^{(\infty;k)} F := \lim_{\xi \to \infty} F(x_1, \ldots, x_{n-k}, \xi \cdot x_{n-k+1}, \ldots, \xi \cdot x_n)$$

The key results on this subspace are summarized in the following theorem:

Thm [FHHSY]:

(a) Suppose $F \in S^{sm}_n$ and $\partial^{(\infty;k)} F$ exist $\forall\, 0 \leq k \leq n \Rightarrow F \in \mathcal{A}^{sm}_n$.

(b) The subspace $\mathcal{A}^{sm}$ is $\overset{sm}{\star}$-commutative.

(c) $\mathcal{A}^{sm}$ is $\overset{sm}{\star}$-closed and it is a polynomial algebra in $\{K_j\}_{j=1}^{\infty}$:

$$K_1(x_1) = x_1^0, \qquad K_m(x_1, \ldots, x_m) = \prod_{i<j} \frac{(x_i - q_1 x_j)(x_j - q_1 x_i)}{(x_i - x_j)^2}$$

Rmk: The proof of this result is actually quite interesting (though simple) and the same ideas were used by Negut to prove $\Psi_1 : \ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)^+ \longrightarrow S$ is an isomorphism.

(2.3) Geometric importance of $\mathcal{A}^{sm}$

Recall the geometric action

$$K_j \in S \subset \ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1) \curvearrowright M = \bigoplus_{n\geq 0} K^{\mathbb{C}^*\times\mathbb{C}^*}((\mathbb{C}^2)^{[n]})$$

On the other hand, due to the localization theorem, M has a distinguished fixed point basis:

$$M = \bigoplus_{\lambda\text{-Young diagram}} F\cdot[\lambda], \quad F := \mathbb{C}(t_1,t_2)$$

Let $\Theta: M \xrightarrow{\sim} \Lambda_F$ be the isomorphism of M and the ring Λ_F of symmetric polynomials in ∞ many variables, defined by

$$[\lambda] \longmapsto P_\lambda^{t_1, 1/t_2}$$ – the Macdonald polynomial

Under this identification, the operator K_j corresponds to a multiplication by $e_j \in \Lambda_F$ – the j^{th} elementary symmetric function.

Applying the identity

$$1 + \sum_{i=1}^{\infty} e_i z^i = \exp\Big(\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} p_i z^i\Big)$$

p_j – power-sum symmetric function

we recover half of the Heisenberg algebra action on M.
To get the opposite half, we repeat the same construction w.r.t. the opposite half $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)^- \subset \ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$.

Rmk: (1) Repeating the same argument in additive case, we recover the classical Nakajima's construction. So we generalized his construction to K-theory

(2) The identification Θ intertwines two canonical pairings:

(i) The pairing on $M = \bigoplus_{n\geq 0} M_n$ is defined by

- $(M_n, M_m) = 0$ if $n \neq m$
- $\forall\, \mathcal{G}_1, \mathcal{G}_2 \in K^{\mathbb{C}^*\times\mathbb{C}^*}((\mathbb{C}^2)^{[n]})_{loc}$ set $(\mathcal{G}_1, \mathcal{G}_2) := [R\Gamma((\mathbb{C}^2)^{[n]}, \mathcal{G}_1 \otimes \mathcal{G}_2 \otimes \det(\mathcal{T})^{-1})]$

(ii) The pairing on Λ_F is the $(q := t_1,\ t := 1/t_2)$ – Macdonald inner product given by $(p_\lambda, p_\mu) = \delta_{\lambda,\mu}\cdot z_\lambda \cdot \prod_{i=1}^{k} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$

(here for $\lambda = (\lambda_1, \ldots, \lambda_k) = (1^{m_1} 2^{m_2} \ldots)$ set $p_\lambda := p_{\lambda_1}\cdot\ldots\cdot p_{\lambda_k}$, $z_\lambda := \prod_{r\geq 1} r^{m_r}\cdot m_r!$)

3.1 The quantum toroidal of sl_n: $\ddot{U}_{q,d}(sl_n)$

Let us now switch to the algebra of main interest in this talk: $\ddot{U}_{q,d}(sl_n)$.
These algebras were first introduced by Ginzburg–Kapranov–Vasserot.

Let:
- $q, d \in \mathbb{C}^*$ - be two parameters
- $(a_{ij})_{i,j=0}^{n-1}$, $(m_{ij})_{i,j=0}^{n-1}$ be defined by $a_{i,i}=2$, $a_{i,i\pm1}=-1$, $m_{i,i\pm1}=\mp1$, other $a_{ij}=m_{ij}=0$.
- the rational f-n $g_m(z) := \frac{q^m z-1}{z-q^m}$
- assume $\boxed{n>2}$

Then, the quantum toroidal algebra of sl_n is generated by

$$\boxed{\{e_{i,k},\ f_{i,k},\ \psi_{i,k},\ \psi_{i,0}^{-1},\ \gamma^{\pm1/2},\ q^{\pm d_1},\ q^{\pm d_2}\}_{0\le i\le n-1}^{k\in\mathbb{Z}}}$$

with the following defining rel-s:

- $[\psi_i^\pm(z), \psi_j^\pm(w)]=0$, $\gamma^{\pm1/2}$ - central
- $\psi_{i,0}^{\pm1}\cdot\psi_{i,0}^{\mp1} = \gamma^{\pm1/2}\cdot\gamma^{\mp1/2} = q^{\pm d_1}\cdot q^{\mp d_1} = q^{\pm d_2}\cdot q^{\mp d_2} = 1$
- $e_i(z)\, e_j(w) = g_{a_{ij}}\left(d^{m_{ij}}\frac{z}{w}\right) e_j(w)\, e_i(z)$
- $f_i(z)\, f_j(w) = g_{a_{ij}}\left(d^{m_{ij}}\frac{z}{w}\right)^{-1} f_j(w)\, f_i(z)$
- $\psi_i^\pm(z)\, e_j(w) = g_{a_{ij}}\left(\gamma^{\pm1/2} d^{m_{ij}}\frac{z}{w}\right) e_j(w)\, \psi_i^\pm(z)$
- $\psi_i^\pm(z)\, f_j(w) = g_{a_{ij}}\left(\gamma^{\mp1/2} d^{m_{ij}}\frac{z}{w}\right)^{-1} f_j(w)\, \psi_i^\pm(z)$
- $\mathrm{Sym}_{\mathfrak{S}_2}\, [e_i(z_1), [e_i(z_2), e_{i\pm1}(w)]] = 0 = \mathrm{Sym}_{\mathfrak{S}_2}\, [f_i(z_1), [f_i(z_2), f_{i\pm1}(w)]]$
- $g_{a_{ij}}\left(\gamma^{-1}\cdot d^{m_{ij}}\frac{z}{w}\right)\psi_i^+(z)\,\psi_j^-(w) = g_{a_{ij}}\left(\gamma d^{m_{ij}}\frac{z}{w}\right)\psi_j^-(w)\,\psi_i^+(z)$
- $q^{d_1} e_i(z) q^{-d_1} = e_i(qz)$, $\quad q^{d_1} f_i(z) q^{-d_1} = f_i(qz)$, $\quad q^{d_1}\psi_i^\pm(z) q^{-d_1} = \psi_i^\pm(qz)$
- $q^{d_2} e_i(z) q^{-d_2} = q\cdot e_i(z)$, $\quad q^{d_2} f_i(z) q^{-d_2} = q^{-1}\cdot f_i(z)$, $\quad q^{d_2}\psi_i^\pm(z) q^{-d_2} = \psi_i^\pm(z)$.

where we use the same notations $e_i(z)$, $f_i(z)$, $\psi_i^\pm(z)$, $\delta(z)$ as for gl_1-case.

<u>Rmk:</u> The el-s $q^{\pm d_1}$, $q^{\pm d_2}$ together with the last two relations are not essential and did not appear in the original definition, but we will need them in order to have the Drinfeld double construction.

(3.2) Big shuffle algebra

Let us now consider another degeneration of the elliptic curve $\mathcal{E}$: into a chain of $\mathbb{P}^1$  (we call it $A_{n-1}^{(1)}$ - type)

Under this degeneration each of $\{\tau_1, \tau_2\}$ degenerates into an automorph. of this chain which is given by one discrete parameter $\in \mathbb{Z}/n\mathbb{Z}$ and n continuous parameters $\in \mathbb{C}^*$. We will consider the simplest case when all continuous parameters $=1$, discrete parameters $\subset \{0, \pm 1\}$.

* * *

As we can't make precise above statements, let us give a rigorous def-n.

- Consider an ambient $\mathbb{N}^n$-graded space $\mathbb{S} = \bigoplus_{\underline{k}=(k_1,\ldots,k_n)} \mathbb{S}_{\underline{k}}$ with $\mathbb{S}_{\underline{k}}$ consisting of rational f-s in $\{x_{i,j} \mid 1 \le i \le n, 1 \le j \le k_i\}$ which are $\mathfrak{S}_{k_1} \times \ldots \times \mathfrak{S}_{k_n}$ - symmetric
- Define the $\star$-product on $\mathbb{S}$ by $\mathbb{S}_{\underline{k}} \times \mathbb{S}_{\underline{\ell}} \xrightarrow{\star} \mathbb{S}_{\underline{k}+\underline{\ell}}$ via

$$(F \star G)\begin{pmatrix} x_{1,1} & & x_{n,1} \\ \vdots & \ldots & \vdots \\ x_{1,k_1+l_1} & & x_{n,k_n+l_n} \end{pmatrix} := \mathrm{Sym}\left[F\begin{pmatrix} x_{1,1} & & x_{n,1} \\ \vdots & \ldots & \vdots \\ x_{1,k_1} & & x_{n,k_n} \end{pmatrix} \cdot G\begin{pmatrix} x_{1,k_1+1} & & x_{n,k_n+1} \\ \vdots & \ldots & \vdots \\ x_{1,k_1+l_1} & & x_{n,k_n+l_n} \end{pmatrix} \cdot \prod_{i,i'} \prod_{j \le k_i}^{j' > k_{i'}} \omega_{i,i'}\left(\frac{x_{i,j}}{x_{i',j'}}\right) \right]$$

This endows $\mathbb{S}$ with a structure of a unital assoc. algebra

$$\omega_{i,i}(t) := \frac{t - q^{-2}}{t-1}, \quad \omega_{i,i+1} := \frac{d^{-1}t - q}{t-1}, \quad \omega_{i,i-1}(t) := \frac{t - qd^{-1}}{t-1}, \quad \omega_{i,j}(t) = 1 \text{ (else)}$$

- The space $\mathbb{S}$ is huge, so we consider $S = \bigoplus_{\underline{k}} S_{\underline{k}}$, $S_{\underline{k}} \subset \mathbb{S}_{\underline{k}}$ given by:
  (i) Pole conditions: $F = \frac{f(x_{1,1}, \ldots, x_{n,k_n})}{\prod_i \prod_{j=1}^{k_i} \prod_{j'=1}^{k_{i+1}} (x_{i,j} - x_{i+1,j'})}$, $f \in \mathbb{C}[x_{i,j}^{\pm 1}]^{\mathfrak{S}_{k_1} \times \ldots \times \mathfrak{S}_{k_n}}$
  (ii) Wheel conditions: $F = 0$ if $x_{i,j_1}/x_{i\pm 1,\ell} = qd^{\pm 1}$, $x_{i\pm 1,\ell}/x_{i,j_2} = qd^{\mp 1}$.

Lemma: S is $\star$-closed.

Def: The algebra $(S, \star)$ is called the big shuffle algebra (of $A_{n-1}^{(1)}$-type).

Thm [Negut]: The natural homomorphism

$$\Psi_n : \ddot{U}_{q,d}(\mathfrak{sl}_n)^+ \longrightarrow S \qquad e_{i,j} \longmapsto x_{i,1}^j$$

is the isomorphism of algebras.

(3.3) Commutative subalgebras $\mathcal{A}(s_1,\ldots,s_n)\subset S$

The key object of this talk is a subspace $\mathcal{A}(s_1,\ldots,s_n)\subset S$ similar to $\mathcal{A}^{sm}$.
For any $0\le \bar{l}\le \bar{k}\in\mathbb{N}^n$, $\xi\in\mathbb{C}^*$ and $F\in S_{\bar{k}}$, we define

$$F_\xi^{\bar{l}}:=F(\xi\cdot x_{1,1},\ldots,\xi\cdot x_{1,l_1},x_{1,l_1+1},\ldots,x_{1,k_1};\ldots;\xi\cdot x_{n,1},\ldots,\xi\cdot x_{n,l_n},x_{n,l_n+1},\ldots,x_{n,k_n})$$

For any integer numbers $a\le b$, we define $\bar{l}:=[a;b]\in\mathbb{N}^n$ by

$$l_i:=\#\{c\in\mathbb{Z}\mid a\le c\le b,\ c\equiv i\ (\mathrm{mod}\ n)\}$$

Key def-n: For any $\bar{s}=(s_1,\ldots,s_n)\in(\mathbb{C}^*)^n$, consider an $\mathbb{N}^n$-graded subspace $\mathcal{A}(\bar{s})=\bigoplus_{\bar{k}\in\mathbb{N}^n}\mathcal{A}(\bar{s})_{\bar{k}}$ defined by

$$\mathcal{A}(\bar{s})_{\bar{k}}=\left\{F\in S_{\bar{k},0}\ \Big|\ \partial^{(\infty;a,b)}F=\prod_{i=a}^{b}s_i\cdot\partial^{(0;a,b)}F\quad\forall\ [a;b]\le\bar{k}\right\}$$

($S_{\bar{k},0}$ means that tot.deg$(F)=0$.)

where $\partial^{(0;a,b)}F:=\lim_{\xi\to 0}F_\xi^{[a;b]}$, $\partial^{(\infty;a,b)}F:=\lim_{\xi\to\infty}F_\xi^{[a;b]}$.

Q-n What can be said about $\mathcal{A}(\bar{s})$?

A certain class of el-s in $\mathcal{A}(\bar{s})$ is described in the following lemma:

Lemma: For any $k\in\mathbb{N}$, $\mu\in\mathbb{C}$, $\bar{s}\in(\mathbb{C}^*)^n$, define $F_k^\mu(\bar{s})\in S_{k,\ldots,k}$ by

$$F_k^\mu(\bar{s}):=\frac{\prod_{i=1}^n\prod_{1\le j\ne j'\le k}(x_{i,j}-q^{-2}x_{i,j'})\cdot\prod_{i=1}^n\left(s_1\cdot\ldots\cdot s_i\prod_{j=1}^k x_{i,j}-\mu\cdot\prod_{j=1}^k x_{i+1,j}\right)}{\prod_{i=1}^n\prod_{j,j'=1}^k(x_{i,j}-x_{i+1,j'})}$$

If $s_1\cdot\ldots\cdot s_n=1$, then $F_k^\mu(\bar{s})\in\mathcal{A}(\bar{s})$

Rmk: (1) We can also look at μ as a formal variable and decompose the above fraction w.r.t. $[\mu^?]$ $\rightsquigarrow$ get another basis of $\mathrm{span}\langle F_k^\mu\mid\mu\in\mathbb{C}\rangle$.

(2) We also get a distinguished el-t

$$F_k:=\frac{\prod_{i=1}^n\prod_{1\le j\ne j'\le k}(x_{i,j}-q^{-2}x_{i,j'})\cdot\prod_i\prod_j x_{i,j}}{\prod_{i=1}^n\prod_{1\le j,j'\le k}(x_{i,j}-x_{i+1,j'})}\in\mathcal{A}(\bar{s})\quad\forall\{s_i\}\text{ s.t. }s_1\cdot\ldots\cdot s_n=1.$$

(it will be used latter on).

(3.4) Main Theorem

Our main result of the recent work joint with Feigin (arXiv: 1504.01696) is the explicit description of $\mathcal{A}(\bar{s})$ for "generic $\bar{s}$".

Theorem [FT']

Assume $s_1 \cdot \ldots \cdot s_n = 1$ and $s_1^{\alpha_1} \cdot \ldots \cdot s_n^{\alpha_n} \in q^{\mathbb{Z}} \cdot d^{\mathbb{Z}} \Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n$. Then:

(a) The space $\mathcal{A}(\bar{s})$ is $\star$-commutative and $\star$-closed

(b) For any pair-wise distinct $\mu_1, \ldots, \mu_n \in \mathbb{C}$, the algebra $\mathcal{A}(\bar{s})$ is a free polynomial algebra in $\{F_k^{\mu_i} \mid k \geq 1, 1 \leq i \leq n\}$.

Idea of the proof

Step 1: Use the Gordon filtration (see [FHHSY] for gl_1-case adapted to sl_n-case by Negut) to obtain the upper bound on $\dim \mathcal{A}(\bar{s})_{\underline{k}}$

Step 2: Show that the subalgebra $\mathcal{A}'(\bar{s})$ generated by all $\{F_{\underline{k}}^{\mu}\}$ sits inside $\mathcal{A}(\bar{s})$.

Step 3: Use another filtration (based on specializations along each $\{x_{i,\cdot}\}$ separately) to deduce the lower bound for $\dim \mathcal{A}'(\bar{s})_{\underline{k}}$, which coincides with upper bound from Step 1.

This implies $\mathcal{A}(\bar{s}) = \mathcal{A}'(\bar{s})$

Step 4: Use the filtration from Step 3 to deduce the commutativity of $\mathcal{A}(\bar{s})$ by induction.

■

Rmk: In particular, we see that for "generic $\bar{s} \in (\mathbb{C}^*)^n$", we have:

$$\boxed{\mathcal{A}(\bar{s}) \subset \bigoplus_{m \geq 0} S_{m \cdot \delta}}, \quad \delta = (1, \ldots, 1) \in \mathbb{N}^n$$

3.5 Vertex-type representations $W(p)_n$ of $\ddot{U}'_{q,d}(sl_n)$

The following construction goes back to Saito (arXiv: 9611030) and generalizes the famous Kac-Frenkel construction for $U_q(\hat{\mathfrak{g}})$.

Settings:
- $\{\bar\alpha_i\}_{i=1}^{n-1}$ – simple roots of sl_n, $\bar Q := \bigoplus_{i=1}^{n-1} \mathbb{Z}\cdot\bar\alpha_i$ – root lattice
- $\{\bar\Lambda_i\}_{i=1}^{n-1}$ – fundamental weights of sl_n, $\bar P := \bigoplus_{i=1}^{n-1}\mathbb{Z}\bar\Lambda_i = \bigoplus_{i=2}^{n-1}\mathbb{Z}\cdot\bar\alpha_i \oplus \mathbb{Z}\bar\Lambda_{n-1}$ – weight lattice
- $\{\bar h_i\}_{i=1}^{n-1}$ – simple coroots of sl_n
- $\bar\alpha_0 := -\sum_{i=1}^{n-1}\bar\alpha_i \in \bar Q$, $\bar\Lambda_0 := 0 \in \bar P$, $\bar h_0 := -\sum_{i=1}^{n-1}\bar h_i$

Define:
- $\mathbb{C}\{\bar P\}$ – the $\mathbb{C}$-algebra generated by $\{e^{\bar\alpha_2},\dots,e^{\bar\alpha_{n-1}},e^{\bar\Lambda_{n-1}}\}$ with the defining relations:
$$e^{\bar\alpha_i}\cdot e^{\bar\alpha_j} = (-1)^{\langle \bar h_i,\bar\alpha_j\rangle} e^{\bar\alpha_j}\cdot e^{\bar\alpha_i}, \quad e^{\bar\alpha_i}\cdot e^{\bar\Lambda_{n-1}} = (-1)^{\delta_{i,n-1}} e^{\bar\Lambda_{n-1}}\cdot e^{\bar\alpha_i}$$
- $\mathbb{C}\{\bar Q\}$ – subalgebra of $\mathbb{C}\{\bar P\}$ generated by $\{e^{\bar\alpha_i}\}_{i=1}^{n-1}$
- For $\alpha = \sum_{i=2}^{n-1} m_i\bar\alpha_i + m_n\bar\Lambda_{n-1}$, we set $e^{\alpha} := (e^{\bar\alpha_2})^{m_2}\cdot\ldots\cdot(e^{\bar\alpha_{n-1}})^{m_{n-1}}\cdot(e^{\bar\Lambda_{n-1}})^{m_n}$.
- Consider the "generalized Heisenberg algebra" S_n generated by $\{H_{i,k} \mid 0\le i\le n-1,\ k\in\mathbb{Z}\setminus\{0\}\}$ and a central el-t H_0 with the defining rel-n $[H_{i,k}, H_{j,l}] = d^{-km_{ij}}\cdot\frac{[k]\cdot[ka_{ij}]}{k}\delta_{k,-l}\cdot H_0$

  Set: $F_n := \mathrm{Ind}_{S_n^{\ge 0}}^{S_n}\mathbb{C}$ – level 1 Fock representation

For every $0\le p\le n-1$ define $W(p)_n := F_n\otimes\mathbb{C}\{\bar Q\}e^{\bar\Lambda_p}$

Define the operators $H_{i,l}, e^{\bar\alpha}, \partial_{\bar\alpha_i}, z^{H_{i,0}}, d : W(p)_n \circlearrowleft$ which act on the vector $v\otimes e^{\bar\beta} = H_{i_1,-k_1}\cdot\ldots\cdot H_{i_N,-k_N}(v_0)\otimes e^{\sum_{j=1}^{n-1} m_j\bar\alpha_j + \bar\Lambda_p}$ by

$$H_{i,l}(v\otimes e^{\bar\beta}) = (H_{i,l}v)\otimes e^{\bar\beta}, \quad e^{\bar\alpha}(v\otimes e^{\bar\beta}) = v\otimes e^{\bar\alpha}e^{\bar\beta}, \quad \partial_{\bar\alpha_i}(v\otimes e^{\bar\beta}) = \langle \bar h_i,\bar\beta\rangle\cdot v\otimes e^{\bar\beta}$$
$$z^{H_{i,0}}(v\otimes e^{\bar\beta}) = z^{\langle\bar h_i,\bar\beta\rangle}\cdot d^{\frac12\sum_{j=1}^{n-1}a_{ij}m_{ij}m_j}\cdot v\otimes e^{\bar\beta}, \quad d(v\otimes e^{\bar\beta}) = \left(-\sum k_i - \tfrac12((\bar\beta,\bar\beta)-(\bar\Lambda_p,\bar\Lambda_p))\right)v\otimes e^{\bar\beta}$$

Thm (Saito): The following f-las define a repr-n of $\ddot{U}'_{q,d}(sl_n)$ on $W(p)_n$ $\forall \bar c\in(\mathbb{C}^*)^n$ (where $\ddot{U}'$ denotes: no $q^{\pm d_2}$ plus we factor by $\psi_{0,0}\cdot\ldots\cdot\psi_{n-1,0}-1$).

$$\rho_{p,\bar c}(e_i(z)) = c_i\cdot\exp\left(\sum_{k=1}^{\infty}\frac{q^{-k/2}}{[k]}H_{i,-k}z^k\right)\cdot\exp\left(-\sum_{k=1}^{\infty}\frac{q^{-k/2}}{[k]}H_{i,k}z^{-k}\right)\cdot e^{\bar\alpha_i}z^{H_{i,0}+1}$$
$$\rho_{p,\bar c}(f_i(z)) = c_i^{-1}\cdot\exp\left(-\sum_{k=1}^{\infty}\frac{q^{k/2}}{[k]}H_{i,-k}z^k\right)\cdot\exp\left(\sum_{k=1}^{\infty}\frac{q^{k/2}}{[k]}H_{i,k}z^{-k}\right)\cdot e^{-\bar\alpha_i}z^{-H_{i,0}+1}$$
$$\rho_{p,\bar c}(\psi_i^{\pm}(z)) = \exp\left(\pm(q-q^{-1})\sum_{k=1}^{\infty}H_{i,\pm k}z^{\mp k}\right), \quad \rho_{p,\bar c}(\gamma^{\pm 1/2}) = q^{\pm 1/2}, \quad \rho_{p,\bar c}(q^{\pm d_1}) = q^{\pm d}$$

3.6 The algebras $\ddot{U}_{q,d}(\mathfrak{sl}_n)$, $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$ as Drinfeld doubles

Recall the notion of the Hopf algebra pairing

Def: For two Hopf algebras A and B with invertible antipods, the map

$$\varphi: A \times B \longrightarrow \mathbb{C}$$

is called the Hopf pairing if it satisfies

$$(1)\quad \varphi(a, bb') = \sum_i \varphi(a_1^{(i)}, b)\, \varphi(a_2^{(i)}, b') \quad \text{for } \Delta(a) = \sum_i a_1^{(i)} \otimes a_2^{(i)}$$

$$(2)\quad \varphi(aa', b) = \sum_i \varphi(a, b_2^{(i)})\, \varphi(a', b_1^{(i)}) \quad \text{for } \Delta(b) = \sum_i b_1^{(i)} \otimes b_2^{(i)}$$

$$(3)\quad \varphi(a, 1_B) = \varepsilon_A(a), \quad \varphi(1_A, b) = \varepsilon_B(b)$$

$$(4)\quad \varphi(S_A(a), b) = \varphi(a, S_B^{-1}(b))$$

Thm: (a) There exists a unique Hopf pairing $\varphi: \ddot{U}^{\geq} \times \ddot{U}^{\leq} \longrightarrow \mathbb{C}$ s.t.

$$\varphi(e_i(z), f_j(w)) = \frac{\delta_{ij}}{q-q^{-1}} \cdot \delta\left(\frac{z}{w}\right) \qquad \varphi(\psi_i^-(z), \psi_i^+(w)) = g_{a_{ij}}\left(d^{m_{ij}} \frac{z}{w}\right)$$

$$\varphi(\gamma^{1/2}, q^{d_1}) = \varphi(q^{d_1}, \gamma^{-1/2}) = q^{-1/2} \qquad \varphi(\psi_{i,0}, q^{d_2}) = \varphi(q^{d_2}, \psi_{i,0}) = q$$

$$\varphi(e_i(z) x^-) = \varphi(x^+, f_i(z)) = 0 \quad \text{for } x^{\pm} = \psi_j^{\mp}(w), \psi_{j,0}^{\pm 1}, \gamma^{1/2}, q^{d_1}, q^{d_2}$$

$$\varphi(\psi_i^-(z), x) = \varphi(x, \psi_i^+(z)) = 1 \quad \text{for } x = \gamma^{1/2}, q^{d_1}$$

$$\varphi(\gamma^{1/2}, q^{d_2}) = \varphi(q^{d_2}, \gamma^{1/2}) = \varphi(\gamma^{1/2}, \gamma^{1/2}) = \varphi(q^{d_i}, q^{d_j}) = 1$$

where $\ddot{U}^{\geq}$ is a (Hopf) subalg. generated by $\{e_{i,k}, \psi_{i,l}, \psi_{i,0}^{\pm 1}, \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}\}_{k \in \mathbb{Z}}^{l \in -\mathbb{N}}$

$\ddot{U}^{\leq}$ is a (Hopf) subalg. generated by $\{e_{i,k}, \psi_{i,l}, \psi_{i,0}^{\pm 1}, \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}\}_{k \in \mathbb{Z}}^{l \in \mathbb{N}}$.

(b) The natural Hopf. alg. homom. $D_\varphi(\ddot{U}^{\geq}, \ddot{U}^{\leq}) \longrightarrow \ddot{U}_{q,d}(\mathfrak{sl}_n)$ induces isom.

$$D_\varphi(\ddot{U}^{\geq}, \ddot{U}^{\leq}) / (x \otimes 1 - 1 \otimes x \mid x = \gamma^{\pm 1/2}, q^{\pm d_1}, q^{\pm d_2}, \psi_{i,0}^{\pm 1}) \xrightarrow{\sim} \ddot{U}_{q,d}(\mathfrak{sl}_n)$$

↑ the classical Drinfeld double defined for any Hopf pairing

(c) "Throwing away" $q^{\pm d_2}$ and setting $\psi_{0,0} := \frac{1}{\psi_{1,0} \cdots \psi_{n-1,0}}$, we get the Drinfeld double construction of $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$ via $D_{\varphi'}(\ddot{U}'^{\geq}, \ddot{U}'^{\leq})$.

(d) If q, qd, qd^{-1} are not roots of 1, then φ, φ' – nondegenerate.

Rmk: The Hopf algebra structure on $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ is due to Ding–Iohara (see also [FT, Thm 1.6] for explicit f-las).

3.7 Drinfeld double, universal R-matrix, transfer matrices

- Generalized Drinfeld double

On the previous page we used the notion of Drinfeld double. Let us recall the general setup. Given Hopf algebras A, B and a Hopf pairing $\varphi: A \times B \longrightarrow \mathbb{C}$ there is a unique Hopf alg. $D_\varphi(A,B)$ s.t.

(1) $D_\varphi(A,B) \simeq A \otimes B$ as coalgebras

(2) Under the natural inclusions $A \overset{a \mapsto a\otimes 1_B}{\hookrightarrow} D_\varphi(A,B) \overset{1_A \otimes b \leftarrow\!\shortmid\, b}{\hookleftarrow} B$, both A and B are Hopf subalgebras of $D_\varphi(A,B)$

(3) For any $a \in A$, $b \in B$, we have:

$$(a\otimes 1)\cdot(1\otimes b) = a \otimes b$$
$$(1\otimes b)\cdot(a\otimes 1) = \sum \varphi(S_A^{-1}(a_1^{(i)}), b_1^{(i)})\cdot \varphi(a_3^{(i)}, b_3^{(i)})\, a_2^{(i)} \otimes b_2^{(i)}$$

- Universal R-matrix

Recall that a Hopf alg. A is called quasitriangular if $\exists$ invertible $R \in A \otimes A$ s.t.

$$R\Delta(x) = \Delta^{op}(x) R\,,\quad (\Delta\otimes \mathrm{Id})(R) = R^{13}R^{23}\,,\quad (\mathrm{Id}\otimes\Delta)(R) = R^{13}R^{12}$$

If $R \in A \hat{\otimes} A$ instead of $A \otimes A$, then we call A formally quasitriangular

Basic Result: If $\varphi: A\times B \longrightarrow k$ is a nondegenerate Hopf pairing, then

$$R := \sum e_i \otimes e_i^* \quad (\{e_i\}\text{ - any basis of } A,\ \{e_i^*\}\text{ - dual w.r.t. } \varphi)$$

is the universal R-matrix of $D_\varphi(A,B)$

- Transfer Matrices

We briefly recall the classical way to construct "large" commutative subalgebras of an appropriate completion $\hat{A}$ if A - formally quasitriangular.

- Fix a group-like element $x \in A$ (or $\hat{A}$), i.e. $\Delta(x) = x \otimes x$
- Given an A-representation V, consider $T_V(x) := (1\otimes \mathrm{tr}_V)((1\otimes x)R)$ if the latter makes sense. ($T_V(x)$ ← transfer matrix)
- Properties of $R \Rightarrow T_\bullet(x)$ defines a homomorphism from a Grothendieck ring of an appropriate subcategory of A-modules to a suitable completion $\hat{A}$.

The image is a commutative subalgebra of $\hat{A}$ and is called the Bethe subalgebra

3.8 The commutative subalgebras $\mathcal{A}(\bar{s})$ vs the Bethe subalgebras

As we said, the algebra $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$ admits a Drinfeld double realization.
Moreover, the defining pairing φ' is nondegenerate if $q, qd, qd^{-1} \neq \sqrt{1}$.

Take a generic Cartan group-like element

$$\boxed{x := u_1^{-\bar{\Lambda}_1} \cdot \ldots \cdot u_{n-1}^{-\bar{\Lambda}_{n-1}} \cdot t^{-d_1}}$$

(we factored by q^{d_2} and $\prod \psi_{i,0}$, while $\gamma^{1/2}$ doesn't affect much since it's central)

Technical Computation

Consider $\ddot{U}'_{q,d}(\mathfrak{sl}_n)$-representations : $\{\rho_{p,\bar{c}} \mid 0 \le p \le n-1,\ \bar{c} \in (\mathbb{C}^*)^n\}$.
Then by general construction we get a commutative family

$$\{T_{\rho_{p,\bar{c}}}(x)\} \text{ or equivalently } \{X_{p,N}(x) =: X_{p,N}^{\bar{u},t}\},$$

where $T_{\rho_{p,\bar{c}}}(x) = \sum_{N=0}^{\infty} \left(\frac{1}{c_0 \ldots c_{n-1}}\right)^N \cdot X_{p,N}(x)$

Straightforward Computation

We can explicitly compute $X_{p,N}^{\bar{u},t}$ (see [FT, Thm 3.8(c)])
What is relevant for our discussion is the limit $X_{p,N}^{\bar{u}} := \lim_{t \to 0} X_{p,N}^{\bar{u},t}$

Thm: $$\boxed{X_{p,N}^{\bar{u}} = \underbrace{\nu_{p,N}}_{\in \mathbb{C}^*} \cdot \frac{\prod_{i=1}^{n} \prod_{1 \le j \ne j' \le N} (x_{i,j} - q^{-2} x_{i,j'})}{\prod_{i=1}^{n} \prod_{1 \le j, j' \le N} (x_{i,j} - x_{i+1,j'})} \cdot (-1)^p [\mu^p] \left\{ \prod_{i=1}^{n} \left(\prod_{j=1}^{N} x_{i+1,j} - \mu u_1 \cdot \ldots \cdot u_{i-1} \prod_{j=1}^{N} x_{i,j} \cdot q^{\bar{\alpha}_i} \right) \right\}}$$

where we use the shuffle realization of $\ddot{U}'^{>}$ obtained just by "adding Cartan loop generators" to S.

Since $\langle \tilde{\delta}, \bar{\alpha}_i \rangle = 0\ \forall i \Rightarrow$ the elements $\{X_{p,N}^{\bar{u}}\}_{p=0}^{n-1}$ are nothing else than the basis elements of span $\langle F_N^{\mu}(\bar{s}) \rangle_{\mu \in \mathbb{C}}$ but with s_i being now not just el-t of $\mathbb{C}^\times$, but rather el-s of $\mathbb{C}^* \cdot e^{\mathbb{C} \otimes_{\mathbb{Z}} \bar{P}}$.

Corollary: In particular, we get an immediate proof of the commutativity of $\{F_k^{\mu}(\bar{s}) \mid k \ge 1, \mu \in \mathbb{C}\}\ \forall \bar{s}$ s.t. $s_1 \cdot \ldots \cdot s_n = 1$.

3.9 The horizontal quantum $U_q(\widehat{gl}_n)$ and its Bethe subalgebras

Classical Construction:

(1) The subalgebra $\dot{U}^v(sl_n) \subset \ddot{U}_{q,d}(sl_n)$ generated by $\{e_{i,k}, f_{i,k}, \psi_{i,k}, \psi_{i,0}^{-1}, \gamma^{\pm 1/2}, q^{\pm d_1} \mid 1 \le i \le n-1, k \in \mathbb{Z}\}$ is isom. to $U_q(\widehat{sl}_n)$

(2) The subalgebra $\dot{U}^h(sl_n) \subset \ddot{U}_{q,d}(sl_n)$ generated by $\{e_{i,0}, f_{i,0}, \psi_{i,0}^{\pm 1}, q^{\pm d_2} \mid 0 \le i \le n-1\}$ is isomorphic to $U_q(\widehat{sl}_n)$ as well.

(3) For every $r \neq 0$, let $\{c_{i,r}\}_{i=0}^{n-1}$ be a nontrivial solution in $\mathbb{C}^n$ of the system $\sum_{i=0}^{n-1} c_{i,r} \cdot [r a_{ij}] d^{-r m_{ij}} = 0$ $(1 \le j \le n-1)$.

Then adding $h_r^v := \sum_{i=0}^{n-1} c_{i,r} \cdot h_{i,r}$ to $\dot{U}^v(sl_n)$ we get $U_q(\widehat{gl}_n)$ inside $\ddot{U}_{q,d}(sl_n)$
(standard choice of Cartan generators via taking $\ln \psi_i^{\pm}(z)$)

<u>Question</u>: Can we also enrich $\dot{U}^h(sl_n)$ to get $\dot{U}^h(gl_n) \simeq U_q(\widehat{gl}_n)$?

<u>Approach 1</u>: Use the beautiful Miki's automorphism $\pi: \ddot{U}_{q,d}(sl_n) \circlearrowleft$

s.t. $\boxed{\dot{U}^v(sl_n) \overset{\pi}{\underset{\pi}{\rightleftarrows}} \dot{U}^h(sl_n),\ \pi: q^{d_1} \mapsto q^{d_2}, q^{d_2} \mapsto q^{-d_1}, \gamma^{1/2} \mapsto \prod_{i=0}^{n-1} \psi_{i,0}, \prod \psi_{i,0} \mapsto \gamma^{-1/2}}$

Then $\dot{U}^h(gl_n)$ is just $\pi(\dot{U}^v(gl_n))$.

<u>Approach 2</u>: Use RTT realization of $U_q(\widehat{gl}_n)$.

This method was used by Negut, who exhibited a shuffle realization of $\dot{U}^h(gl_n)^+$. He proved that under the isom. $\Psi_n: \ddot{U}_{q,d}(sl_n)^+ \xrightarrow{\sim} S$, the subalgebra $\dot{U}^h(gl_n)^+$ is getting identified with $\mathcal{A} = \bigoplus_{\underline{k}} \mathcal{A}_{\underline{k}}$, where

$$\boxed{\mathcal{A}_{\underline{k}} = \{F \in S_{\underline{k},0} \mid \exists \lim_{\xi \to \infty} F_\xi^{\underline{\ell}} \ \forall\, 0 \le \underline{\ell} \le \underline{k}\}}$$

while the positive generators of additional Heisenberg (kind of $h_{r}^h, r>0$) are uniquely (up to a nonzero constant) characterized by:

$$\boxed{X_r \in S_{r\delta,0} \text{ and } \lim_{\xi \to \infty} (X_r)_\xi^{\underline{\ell}} = 0 \ \ \forall\, 0 < \underline{\ell} < r\delta}$$

<u>Our main thm</u> $\Rightarrow$ $\boxed{\begin{array}{l}\text{(i) For "generic" } \bar{s} \in (\mathbb{C}^*)^n:\ \mathcal{A}(\bar{s}) \subset \dot{U}^h(gl_n)^+ \\ \text{(ii) Up to constant } X_r \text{ coincides with } [t^r]\{\ln(\sum_{k=0}^{\infty} F_k \cdot t^k)\}\end{array}}$

<u>Final Thm</u>: Since the level 0 part of $W(p)_n$ is $\dot{U}^h(Lgl_n)$-invariant and $\simeq L_q(\bar{\Lambda}_p)$, we see that the whole Bethe subalgebra of $\dot{U}^h(Lgl_n)$ for $x = u_1^{-\bar{\Lambda}_1} \cdots u_{n-1}^{-\bar{\Lambda}_{n-1}}$ is just the algebra $\mathcal{A}(\bar{s})$ with appropriate $s_i \in \mathbb{C}^* \cdot e^{\text{coop}}$.

3.10 gl_1-case revisited

Actually all our constructions could be also applied to $\ddot{U}_{q_1,q_2,q_3}(gl_1)$.
On one side we have representations $\mathcal{F}(u)$ with $\gamma^{\pm 1/2}$ acting as $Id_{\mathcal{F}(u)}$.
On the other hand, one can also construct a simple vertex-type representation $\{W_c\}_{c\in\mathbb{C}}$ with $\psi_0^{\pm 1}$ acting as identity, while $\gamma^{1/2} \mapsto q_3^{1/4}\cdot Id_{W_c}$.

Applying the same algorithm as before we get el-s
$T_{W_c}(\underline{t}^{d_1/2})$ - transfer matrices of "generic Cartan el-t."

Decomposing w.r.t. powers of c^{-1} and letting $t\to 0$, we get exactly the generators K_N of the commutative algebra

$$\boxed{\mathcal{A}^{sm}\subset \mathcal{S}^{sm}\simeq \ddot{U}_{q_1,q_2,q_3}(gl_1)^+}$$

! This recovers a new incarnation of the commut. subalgebra $\mathcal{A}^{sm}$.

Rmks: (1) Similarly to Miki's automorphism π of $\ddot{U}_{q,d}(sl_n)$, the algebra $\widetilde{\ddot{U}}_{q_1,q_2,q_3}(gl_1)$ (with $q_3^{\pm d_1/2}, q_3^{\pm d_2/2}$ added) admits also a similar automorphism due to elliptic Hall realization of it [Schiffmann-Vasserot]

$$\boxed{\begin{aligned}\pi:\ & e_0\mapsto h_{-1},\ h_{-1}\mapsto f_0,\ f_0\mapsto h_1,\ h_1\mapsto e_0\\ & d_1\mapsto -d_2,\ d_2\mapsto d_1,\ \gamma^{1/2}\mapsto \psi_0^{-1},\ \psi_0\mapsto \gamma^{1/2}\end{aligned}}$$

This automorphism intertwines the Fock representations $\{\mathcal{F}(u)\}$ and the vertex-type representations $\{W_c\}$.

(2) In the case of $\ddot{U}_{q,d}(sl_n)$, Feigin-Jimbo-Miwa-Mukhin also constructed completely analogous representations $V^{(p)}(u)$, $\mathcal{F}^{(p)}(u)$ $(0\le p\le n-1)$, where $\gamma^{\pm 1/2}$ acts by identity.

Again these repr-s $\{\mathcal{F}^{(p)}(u)\}$ and Saito's representations $\{\rho_{p,\bar{c}}\}$ are intertwined by Miki's automorphism π.